13/01/20

$(1-x^2)y''-2xy'+p(p+1)y=0$ (Legendre)

$(1-x^2)y''-xy'+p^2y=0$ (chebyshev)

$y''-2xy'-2py=0$ (hermie)

$xy''+(1-x)y'+py=0$ (laguerre)

$x^2y''+xy'+(x^2-p^2)y=0$ (bessel)

$\int w(x) P_m P_n \, dx = 0$

Legerdre $(1-x^2)y''-2xy'+p(p+1)y=0 \begin{cases} 1 \\ x_0=0 \text{ ομαλό} \end{cases}$

$$A_1(x)=\frac{\alpha_1(x)}{\alpha_2(x)}(x-1)=\frac{-2x}{(1-x)(1+x)}(x-1)=\frac{2x}{1+x}$$

$$y(x)=\sum_{n=0}^{\infty} c_n x^n$$

$$(1-x^2)\left[\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}\right]-2x\left[\sum_{n=0}^{\infty} c_n n x^{n-1}\right]+p(p+1)\sum_{n=0}^{\infty} c_n x^n=0$$

$$\sum_{n=-2} (n+2)(n+1)c_{n+2}x^n=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

$$c_{n+2}=\frac{-(p-n)(p-n+1)}{(n+1)(n+2)}c_n, \quad n=0,1.$$

$n=2k \quad k=0,1$

$$c_{2(k+1)}=\frac{-(p-2k)(p+2k+1)}{(2k+1)(2k+2)}c_{2k}$$

$k=0 \quad C_2 = \dfrac{-p(p+1)}{2} C_0$

$k=1 \quad C_4 = \dfrac{-(p-2)(p+3)}{3\cdot 4} C_2$

$k=2 \quad C_6 = \dfrac{-(p-4)(p+5)}{5\cdot 6} C_4$

$\vdots$

$k=n-1 \quad C_{2n} = \dfrac{-(p-2n+2)(p+2n-1)}{(2n-1)2n} C_{2n-2}$

$$\Rightarrow C_{2n} = (-1)^n \frac{(p-2n+2)\cdots(p-2)p(p+1)(p+3)\cdots(p+2n-1)}{[1\cdot 3\cdot 5\cdots(2n-1)][2\cdot 4\cdot 6\cdots(2n)]} C_0$$

$y_1(x) = \sum_{k=0}^{\infty} C_{2k} x^{2k}, \quad y_2(x) = \sum_{k=0}^{\infty} C_{2k+1} x^{2k+1}$

$p=10: \quad C_6 = 0 \quad ((p-2\cdot 6+2)=0)$

$\sum_{n=0}^{5} C_{2n} x^{2n}$

$p = m \in \mathbb{N} \rightsquigarrow P_m(x)$ m τάξης

$\rightsquigarrow [(1-x^2)P_m']' + m(m+1)P_m = 0$

$m, n \in \mathbb{N}, \; m \neq n$

$[(1-x^2)P_n']' + n(n+1)P_n = 0$

$\Rightarrow [(1-x^2)P_m']'P_n - [(1-x^2)P_n']'P_m + [m(m+1) - n(n+1)]P_m P_n = 0$

$\Rightarrow \int_{-1}^{1} \{[(1-x^2)P_m']'P_n - [(1-x^2)P_n']'P_m\} dx + [m(m+1)-n(n+1)]\int_{-1}^{1} P_m P_n$

$$(1-x^2)P_m' P_n \Big|_{x=-1}^{x=1} - \int_{-1}^{1} (1-x^2) P_n' P_m' \, dx - (1-x^2) P_n' P_m \Big|_{x=-1}^{x=1} + \int_{-1}^{1} (1-x^2) P_n' P_m' \, dx$$

<u>Θεωρώ τη συνάρτηση</u>: $\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = 0$

$\rightsquigarrow [p(x)y']' + \lambda q(x) y_p = 0.$ $\quad \alpha, b \in I \subseteq D_\lambda$

$p(a) = p(b) = 0$

$\lambda_1 \neq \lambda_2 \rightsquigarrow y_1, y_2$

$[p y_1'] + \lambda_1 q y_1 = 0$ ως y_2

$[p y_2'] + \lambda_2 q y_2 = 0$ ως y_1

$$\int_a^b \{ [p y_1']' y_2 + \lambda_1 y_1 y_2 q - [p y_2']' y_1 - \lambda_2 q y_1 y_2 \} dx = 0$$

$$(p y_1') y_2 \Big|_a^b - (p y_2') y_1 \Big|_a^b - (\lambda_1 - \lambda_2) \int_a^b q y_1 y_2 \, dx = 0.$$

$(1-x^2) y'' - x y' + p^2 y = 0 \quad (x_0 = 0)$

0

$$y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0$$

$$e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$\left[\frac{1}{\sqrt{1-x^2}} y' \right]' + \frac{p^2}{\sqrt{1-x^2}} y = 0$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x) dx = 0$$

γενικευμένο $\displaystyle\int_0^\infty f(x)dx = \lim_{k \to +\infty} \int_0^k f(x) dx$

$$\lim \int_{-c}^{c} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx = c$$

$y'' - 2xy' - 2py = 0$, $x_0 = 0$, $p \in \mathbb{N}$.

$$e^{-\int 2x\,dx} = e^{-x^2}$$

$$e^{-x^2} y'' - 2x e^{-x^2} y' - 2p e^{-x^2} y = 0$$

$$\left[e^{-x^2} y'\right]' - 2p e^{-x^2} y = 0$$

H_m, H_n , $m, n \in \mathbb{N}$, $m \neq n$

$\int e^{-x^2} H_m(x) H_n(x)\, dx$ μηδενίζ μόνο στο $\pm\infty$.

$$\to \int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x)\, dx$$

$$\to \lim_{c \to \infty} \int_{-c}^{c} e^{-x^2} H_m(x) H_n(x)\, dx$$

$$y'' + \frac{(1-x)}{x} y' + \frac{py}{x} = c$$

$$e^{\int \frac{1-x}{x} dx} = e^{\log x - x} = x \cdot e^{-x}$$

L_m, L_n

$$\int_0^{+\infty} \frac{1}{x} L_m(x) L_n(x)\, dx.$$

$\lim_{\lambda \to +\infty} \int_{1/\lambda}^{\lambda} \dots$ ή $\int_0^1 \dots + \lim_{x \to +\infty} \int_1^x$

$\cdot\ x^2y'' + xy' + (x^2-p^2)y = 0$

$xy'' + y' + (x - \frac{p^2}{x})y = 0$

$(xy')' + (x - \frac{p^2}{x})y = 0$ ($\lambda_0 = 0$ ανώμαλο σημείο)

$J_p(\lambda) \rightsquigarrow \lambda_0 < \lambda_1 < \lambda_2 < \dots \to +\infty.$

$u(x) = J_p(\lambda x)$

$u(0) = 0$

$u(1) = 0$

$u'(x) = \lambda J_p'(\lambda x)$

$u''(x) = \lambda^2 J''_p(\lambda x)$

$\rightsquigarrow \lambda x \quad \lambda^2 x^2 J''_p(\lambda x) + \lambda x J'_p(\lambda x) + (\lambda^2 x^2 - p^2) J_p(\lambda x) = 0.$

$x^2 J_p' + x J_p' + (x^2 - p^2) J_p = 0$

$x^2 u''(x) + x u'(x) + (\lambda^2 x^2 - p^2) u(x) = 0$

$x u''(x) + u'(x) + (\lambda^2 x - \frac{p^2}{x}) u(x) = 0$

$[x u'(x)]' + (\lambda^2 x - \frac{p^2}{x}) u(x) = 0$

$\int_0^1 x J_p(\lambda_1 x) J_p(\lambda_2 x)\, dx = 0$